(7 pages) **Reg. No. :** ....

Code No. : 5088 **Sub. Code : HMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2020.

Fourth Semester

Mathematics

FUNCTIONAL ANALYSIS

(For those who joined in July 2012-2015)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Every linear transformation of N into an arbitrary normal linear space N' is ——————.

   (a) Bounded

   (b) Unbounded

   (c) Continuous

   (d) Discontinuous

2. If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image $T(S)$ is a ——————set in N^1.

(a) Bounded (b) Unbounded

(c) Continuous (d) Discontinuous

3. If x and y are any two vectors in a Hilbert space, then $|(x, y)| - \|x\|\|y\|$

(a) $\leq$ (b) $<$

(c) $\geq$ (d) $>$

4. If M and N are closed linear spaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M + N$ is ——————.

(a) open (b) closed

(c) union (d) disjoint

5. Which one is the property of orthonormal?

(a) $i = j \Rightarrow e_i \perp e_j$ (b) $\|e_i\| = 0$ for every i

(c) $\|e_i\| = 1$ for every i (d) $i = j \Rightarrow e_i \| e_j$

6. Every non-zero Hilbert space contains a complete ———————set.

(a) parallel (b) normal

(c) orthonormal (d) closed

7. If N is a normal operator on H, then $\left\|N^2\right\| =$ ———————

(a) 1 (b) 0

(c) N (d) $\|N\|^2$

8. The unitary operators on H form a ———————.

(a) subgroup (b) cyclic subgroup

(c) group (d) abelian group

9. If A is a division algebra, then it equals the set of all scalar multiples of the ———————.

(a) identity (b) constant

(c) reciprocal (d) inverse

10. If G is an open set, then S is a ——————— set.

(a) open (b) closed

(c) normal (d) orthonormal

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).
Each answer should not exceed 250 words.

11. (a) Prove that if N is a normal linear space and x_0 is a non-zero vector in N, then there exist a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

Or

(b) If N and N' be the normal linear spaces and T a linear transformation of N into N'. Then prove that the following conditions on T are all equivalent : (i) T is continuous (ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

12. (a) State and prove open mapping theorem.

Or

(b) Prove that if M is a closed linear space of a Hilbert space H, then $H = M \oplus M^{\perp}$.

13. (a) Prove that an operator T on H is self adjoint $\Leftrightarrow (T_x, x)$ is real for all x.

Or

(b) Prove that if $\{e_i\}$ is an orthonormal set in a Hilbert space H, and if x is any vector in H, then the set $S = \{e_i : (x_i e_i) \neq 0\}$ is either empty or countable.

14. (a) Prove that if P is the projection on a closed linear subspace M of H, then M is invariant under operator $T \Leftrightarrow TP = PTP$.

Or

(b) Prove that if T is normal, then the M_i's are pairwise orthogonal.

15. (a) Prove that G is an open sat, and therefore S is a closed set.

Or

(b) Prove that $\sigma(x^n) = \sigma(x)^n$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)
Each answer should not exceed 600 words.

16. (a) Prove that let M be a closed linear subspace of a normal linear space N. If the norm of coset $x + M$ in the quotient space N / M is defined by $\|x + M\| = \inf f\{\|x + M\| : m \in M\}$. Then N/M is a normal linear space

Or

(b) Show that let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0 such that $\|f_0\| = \|f\|$.

17. (a) State and prove closed graph theorem.

Or

(b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

18. (a) State and prove Bessel's Inequality.

Or

(b) Prove that let H be a Hilbert space, and let f be an arbitrary functional in H^*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

19. (a) Prove that if P is a projection on H with range M and null space N, then $M \perp N \Leftrightarrow P$ is self adjoint; and in this case $N = M^{\perp}$.

Or

(b) Prove that if T is normal, then the M_i's span H.

20. (a) Prove that if r is an element of A with the property that $1 - xr$ is regular for every x, then r is in R.

Or

(b) Prove that $r(x) = \lim \|x^n\|^{1/n}$.